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Thermal Convection in a Magnetised Conducting Fluid with the Cattaneo-Christov Heat-Flow Model

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By substituting the Cattaneo-Christov heat-flow model for the more usual parabolic Fourier law, we consider the impact of hyperbolic heat-flow effects on thermal convection in the classic problem of a magnetised conducting fluid layer heated from below. For stationary convection the system is equivalent to that studied by Chandrasekhar [*Hydrodynamic and Hydromagnetic Stability* (1981)], and with free boundary conditions we recover the classical critical Rayleigh number $R_c^{(c)}(Q)$ which exhibits inhibition of convection by the field according to $R_c^{(c)} \to \pi^2 Q$ as $Q \to \infty$, where Q is the Chandrasekhar number. However, for oscillatory convection we find that the critical Rayleigh number $R_c^{(o)}(Q, \mathcal{P}_1, \mathcal{P}_2, C)$ is given by a more complicated function of the thermal Prandtl number $\mathcal{P}_1$, magnetic Prandtl number $\mathcal{P}_2$, and Cattaneo number C. To elucidate features of this dependence we neglect $\mathcal{P}_2$ (in which case overstability would be classically forbidden), and thereby obtain an expression for the Rayleigh number that is far less stongly inhibited by the field, with limiting behaviour $R_c^{(o)} \to \pi\sqrt{Q}/C$, as $Q \to \infty$. One consequence of this weaker dependence is that onset of instability occurs as overstability provided C exceeds a threshold value $C_T(Q)$; indeed, crucially we show that when Q is large, $C_T \propto 1/\sqrt{Q}$, meaning that oscillatory modes are preferred even when C itself is small. Similar behaviour is demonstrated in the case of fixed boundaries by means of a novel numerical solution.

1. Introduction

It has long been known that the parabolic form of the classical Fourier heat-flow law presents a physical paradox whereby thermal disturbances are predicted to propagate with infinite speed [2,14,15,19]. Several approaches have been proposed to address this apparent

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inconsistency (including, for example, ballistic [6,21,34], and relativistic heat-transfer equations [23]); however, perhaps the most commonly encountered model is the Maxwell-Cattaneo formulation [2,24][1] whereby the heat-flow Q_i is expressed in terms of gradients in the local temperature T and thermal conductivity κ (i.e., the classical Fourier law), combined with an inertial term accounting for some thermal relaxation time τ, that is,

$$\tau\frac{\partial Q_i}{\partial t}+Q_i=-\kappa\frac{\partial T}{\partial x_i}. \tag{1.1}$$

Crucially, the introduction of an inertial term yields a hyperbolic thermal-energy equation (cf. equation (2.1*b*)), thereby removing the infinite-speed paradox by allowing thermal disturbances to propagate with finite velocity as heat-waves [19] ('second-sound' [11,14,15]). Note that phenomena introduced by the inertial term in τ are often best described by the *Cattaneo number* $C(\tau)$, a dimensionless parameter which accounts for combined system and material properties, while Christov & Jordan [7] have shown that in moving media a convective derivative should be employed, a formulation we refer to as *Cattaneo-Christov* heat-flow (see equations (2.9) and (2.1*c*)).

Hyperbolic heat-flow effects have a range of practical applications that extend beyond their foundational significance. For example, thermal waves are important in the study of thermal transport in nanomaterials and nanofluids [18,21], and thermal shocks in solids [33], and for heat transport in biological tissue and surgical operations [9,22,23,32]. Similarly, thermal relaxation has been shown to impact on flow velocity profiles in Jeffrey fluids [12], and a number of thermal convection problems in fluids and porous media [27,29–31] (including thermo-haline convection [13,28]), while type-II flux laws analogous to equation (1.1) have found utility in related contexts involving advection-diffusion systems [16,17,26].

In accordance with these varied applications, especially those involving convection (see, e.g., Straughan *et al.* [27–31]), recently we studied the impact of the Cattaneo-Christov heat-flow model on the canonical Rayligh-Bénard problem of a Boussinesq fluid layer heated from below [1,3,25]. Our analysis in this earlier context showed that in addition to onset of instability by stationary convection (as predicted classically [3,25]), Cattaneo effects give rise to oscillatory convection as the preferred manner of instability whenever C exceeds some threshold value C_T, where $C_T(\mathcal{P}_1)$ is a function of the Prandtl number $\mathcal{P}_1$ [1]. Here we develop these ideas to study hyperbolic heat-flow effects on convection in a fluid layer subject to an impressed magnetic field. Such an investigation is desirable for at least two reasons: first, magnetic fields are known to strongly suppress the onset of instability in the classical (Fourier law) Bénard problem [4,5], and would therefore be expected to impact on convection with the Cattaneo-Christov heat-flow; and second, introduction of an impressed field in the classical problem results in overstability whenever the Chandrasekhar number Q exceeds a threshold value $Q_T(\mathcal{P}_1,\mathcal{P}_2)$, where $\mathcal{P}_2$ is the magnetic Prandtl number [20], suggesting potential for interaction between magnetic field induced [4,5], and Cattaneo induced [1] overstability once hyperbolic effects are accounted for.

While the problem of thermal convection in a magnetised fluid with the Cattaneo-Christov heat-flow model is readily formulated in terms similar to our earlier study [1], the inclusion of magnetic field effects is non-trivial, and requires substantial theoretical development. Here we begin by introducing the augmented thermal convection model (§2), and consider the problem of stationary convection (§3), in which case we find that hyperbolic heat-flow effects do not impact on the critical Rayleigh number $R_c^{(c)}$ for onset of instability, and we recover Chandrasekhar's classical result obtained using the Fourier heat-flow [3]: i.e., strong inhibition of convection by the impressed field, with asymptotic behaviour $R_c^{(c)}(Q)\to\pi^2 Q$ as $Q\to\infty$. The problem of oscillatory convection subject to free boundary conditions is then considered in §4, where we derive general solutions for both the Rayleigh number $R_a^{(o)}(a)$, and oscillation frequency $\gamma(a)$, at given wavenumber a. As expected, onset of oscillatory convection is found to be determined by all four parameters $\mathcal{P}_1$, $\mathcal{P}_2$, Q and C, leading to an expression for the Rayleigh number

[1] Indeed, some apparently novel heat-flow formulations, such as the ballistic model by Xu and Hu [34], are equivalent to the Maxwell-Cattaneo law (see, e.g., Christov & Jordan [8]).

$R_a^{(o)}(a;\mathcal{P}_1,\mathcal{P}_2,Q,C)$ in which the various dependencies are somewhat obscured. Nevertheless, following in the tradition of Chandrasekhar [3], analytical progress can be made by neglecting the magnetic Prandtl number $\mathcal{P}_2$, meaning overstability would be classically forbidden [3]. Indeed, by proceeding in this way, expressions for both the critical Rayleigh number $R_c^{(o)} \equiv R_c^{(o)}(a_c)$, and the corresponding critical wavenumber a_c are derived in section §4(b), while conditions for permitted solutions are explored in §4(a). Crucially, we show that the critical Rayleigh number for oscillatory convection scales as $R_c^{(o)}(Q) \to \pi\sqrt{Q}/C$ for $Q \to \infty$, that is, much weaker suppression of instability by the impressed field when compared to stationary convection. Thus, for given Chandrasekhar number Q, we expect some threshold Cattaneo number C_T beyond which $(C > C_T)$ the preferred manner of onset of instability switches from stationary to oscillatory convection; here we find that $C_T \propto 1/\sqrt{Q}$ for large Q, meaning that overstability can be preferred even if the Cattaneo number itself is relatively small §4(c). For fixed boundary conditions we study the problem using a novel numerical scheme to compute threshold values for Prandtl numbers $\mathcal{P}_1 = 1$ and $\mathcal{P}_1 = 10$, and Chandrasekhar numbers in the range $Q \in [10^{-1}, 10^{+4}]$, thereby obtaining qualitatively similar behaviour to that found analytically for free boundaries (§5). The nature of transitions between stationary and oscillatory convection, which are characterised by discontinuous shifts in the critical wavenumber a_c, are discussed further in Section 6.

2. Thermal Convection Model

To extend our earlier model [1], the fluid equations governing conservation of mass and energy, alongside Christov's Galilean invariant formulation of the Cattaneo heat-flow law [2,7], are augmented to account for magnetic field evolution (cf. Chandrasekar [3]). Hence, denoting the velocity, pressure, temperature, heat-flow, and magnetic field as v_i, P, T, Q_i, and H_i respectively, the momentum, energy, heat-flow, and induction equations comprising our basic model are

$$\left(\frac{\partial v_i}{\partial t} + v_j\frac{\partial v_i}{\partial x_j}\right) = \frac{\mu H_j}{4\pi\rho_0}\frac{\partial H_i}{\partial x_j} - \frac{\partial}{\partial x_i}\left(\frac{P}{\rho_0} + \frac{\mu H^2}{8\pi\rho_0}\right) + \nu\nabla^2 v_i - \frac{\rho}{\rho_0}g\lambda_i, \tag{2.1a}$$

$$\rho_0 c_V\left(\frac{\partial T}{\partial t} + v_j\frac{\partial T}{\partial x_j}\right) = -\frac{\partial Q_j}{\partial x_j}, \tag{2.1b}$$

$$\tau\left(\frac{\partial Q_i}{\partial t} + v_j\frac{\partial Q_i}{\partial x_j}\right) = -Q_i - \kappa\frac{\partial T}{\partial x_i}. \tag{2.1c}$$

$$\left(\frac{\partial H_i}{\partial t} + v_j\frac{\partial H_i}{\partial x_j}\right) = H_j\frac{\partial v_i}{\partial x_j} + \eta\nabla^2 H_i, \tag{2.1d}$$

where the fluid viscosity ν, gravitational acceleration $g = |\mathbf{g}|$, specific heat c_V, thermal relaxation time τ, thermal conductivity κ, magnetic permeability μ, and resistivity η are constant coefficients. As in our earlier study [1], we assume a Cartesian (x, y, z) geometry in which gravity acts in the negative z-direction, such that λ_i is the unit vector $\lambda_i = (0, 0, 1)$, while the Laplacian operator is $\nabla^2 \equiv \sum_i \partial^2/\partial x_i^2$. The model is closed by an incompressible equation of state, alongside Maxwell's expression for the divergence-free magnetic field, *viz*

$$\frac{\partial v_i}{\partial x_i} = 0, \quad \text{and} \quad \frac{\partial H_i}{\partial x_i} = 0, \tag{2.2}$$

in addition to the standard Boussinesq approximation in the buoyancy term, namely

$$\rho(T) = \rho_0\left[1 + \alpha(T_\alpha - T)\right], \tag{2.3}$$

where ρ_0 is the fluid density when it is at temperature $T = T_\alpha$, and α is a thermal expansion coefficient.

Supposing the fluid to be confined within the semi-infinite region $(x, y) \in \mathbb{R}^2$ between parallel planes $z = 0$ and $z = d$, with upper and lower planes held at fixed temperatures T_u and T_l

respectively, our boundary conditions are

$$v_z = w = 0 \quad \text{at} \quad z = 0, d, \quad \text{and} \quad T(0) = T_l \quad \text{and} \quad T(d) = T_u, \quad \text{where} \quad T_l > T_u. \tag{2.4}$$

Thus, assuming an impressed uniform magnetic field H aligned with the z-axis, the steady-state conducting solution to system (2.1), which we denote using the subscript '0', is

$$v_{i0} = 0, \quad H_{i0} = H\lambda_i, \quad T_0 = T_l - \beta\lambda_j x_j, \quad \text{and} \quad Q_{i0} = \beta\kappa\lambda_i, \quad \text{with} \quad \beta = \frac{T_l - T_u}{d}, \tag{2.5}$$

where β represents the temperature gradient, and buoyancy is balanced by gradients in the pressure $P = P_0(z)$, i.e., $\frac{\partial P_0}{\partial x_i} = -g\rho(T_0)\lambda_i$. To study the stability of the steady-state conducting solution (2.5), we add a set of perturbations $\{u_i, \theta, q_i, p, h_i\}$ such that

$$v_i = v_{i0} + u_i = u_i, \quad T = T_0 + \theta, \quad Q_i = Q_{i0} + q_i, \quad P = P_0 + p, \quad \text{and} \quad H_i = H\lambda_i + h_i, \tag{2.6}$$

and then derive equations governing the perturbed quantities normalised according to

$$\tilde{x}_i = \frac{x_i}{d}, \quad \tilde{t} = \frac{\nu t}{d^2}, \quad \tilde{v}_i = \frac{v_i d}{\nu}, \quad \tilde{P} = \frac{P d^2}{\nu^2 \rho_0}, \quad \tilde{H}_i = \frac{H_i}{H}, \quad \tilde{T} = \sqrt{\frac{\alpha g \kappa d^2 T^2}{\beta \nu^3 \rho_0 c_V}}, \quad \tilde{Q}_i = \frac{\tilde{T} Q_i d}{T\kappa}. \tag{2.7}$$

Indeed, in this way we obtain a fully non-linear dimensionless system of perturbation equations which can be linearised to give (cf. Bissell [1] and Chandrasekar [3])

$$\frac{\partial}{\partial t}\nabla^2 w = R\left(\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2}\right) + \frac{Q}{\mathcal{P}_2}\frac{\partial}{\partial z}\nabla^2 h_z + \nabla^4 w, \tag{2.8a}$$

$$\mathcal{P}_1\frac{\partial\theta}{\partial t} = Rw - s_q, \quad \text{with} \quad s_q = \frac{\partial q_i}{\partial x_i}, \tag{2.8b}$$

$$2C\mathcal{P}_1\frac{\partial s_q}{\partial t} = -s_q - \nabla^2\theta, \tag{2.8c}$$

$$\mathcal{P}_2\frac{\partial h_z}{\partial t} = \nabla^2 h_z + \mathcal{P}_2\frac{\partial w}{\partial z}, \tag{2.8d}$$

where the tilde notation has been dropped for brevity, $w = u_z$ and h_z are the z-components of the velocity and magnetic field perturbations respectively, and we have made use of the dimensionless thermal and magnetic Prandtl numbers, $\mathcal{P}_1$ and $\mathcal{P}_2$ respectively, Cattaneo number C, Chandrasekhar number Q, and Rayleigh number $R_a = R^2$, i.e.,

$$\mathcal{P}_1 = \frac{\nu\rho_0 c_V}{\kappa}, \quad \mathcal{P}_2 = \frac{\nu}{\eta}, \quad Q = \frac{\mu H^2 d^2}{4\pi\rho_0\nu\eta}, \quad C = \frac{\tau\kappa}{2\rho_0 d^2 c_V}, \quad \text{and} \quad R_a = R^2 = \frac{\alpha g d^4\beta\rho_0 c_V}{\nu\kappa}. \tag{2.9}$$

Proceeding with the usual analysis, the perturbations are then decomposed into normal modes based on eigenfunctions W, Θ, S, and K, and an exponential time dependence $\propto e^{\sigma t}$, with σ as a constant frequency, thus:

$$w = W(z)f(x,y)e^{\sigma t}, \quad \theta = \Theta(z)f(x,y)e^{\sigma t}, \quad s_q = S(z)f(x,y)e^{\sigma t}, \quad h_z = K(z)f(x,y)e^{\sigma t}, \tag{2.10}$$

where $f(x,y)$ is a plane tiling function satisfying $\nabla^2 f(x,y) = -a^2 f(x,y)$, with a as a characteristic wavenumber or inverse length-scale. System (2.8) may then be expressed as

$$\sigma(D^2 - a^2)W = -a^2 R\Theta + (D^2 - a^2)^2 W + \frac{Q}{\mathcal{P}_2}D(D^2 - a^2)K, \tag{2.11a}$$

$$\sigma\mathcal{P}_1\Theta = RW - S, \tag{2.11b}$$

$$(2C\mathcal{P}_1\sigma + 1)\, S = -(D^2 - a^2)\Theta, \tag{2.11c}$$

$$\sigma\mathcal{P}_2 K = (D^2 - a^2)K + \mathcal{P}_2 DW, \tag{2.11d}$$

where D is the ordinary differential gradient operator in the z-direction, i.e., $D \equiv \frac{\mathrm{d}}{\mathrm{d}z}$, $D^2 \equiv \frac{\mathrm{d}^2}{\mathrm{d}z^2}$. System (2.11) may be further reduced by eliminating Θ, S, and K to obtain the eighth-order eigenfunction problem in W forming the basis of our subsequent analysis

$$(D^2-a^2)\left[(D^2-a^2)-\mathcal{P}_1\sigma(2C\mathcal{P}_1\sigma+1)\right]\left[(D^2-a^2-\sigma)(D^2-a^2-\sigma\mathcal{P}_2)-QD^2\right]W$$
$$=-a^2R^2(2C\mathcal{P}_1\sigma+1)(D^2-a^2-\sigma\mathcal{P}_2)W. \tag{2.12}$$

Note that here we shall consider two sets of boundary conditions, both with the assumption of perfect conduction (in which case the magnetic field perturbation vanishes at $z=0,1$): free surfaces (no tangental stress); and fixed sufaces (no slip), that is (see Chandrasekhar [3]):

$$\text{free surfaces:}\quad W=0,\quad D^2W=0,\quad \Theta=0,\quad \text{and}\quad K=0,\quad \text{at}\quad z=0,1, \tag{2.13a}$$

$$\text{fixed surfaces:}\quad W=0,\quad DW=0,\quad \Theta=0,\quad \text{and}\quad K=0,\quad \text{at}\quad z=0,1, \tag{2.13b}$$

where the former admits analytical discussion, and the latter must be investigated numerically.

3. Stationary Convection

In the case of stationary convection, onset of instability in system (2.10) occurs through the marginal state $\sigma=0$, for which the eigenfunction problem of equation (2.12) reduces to

$$\left(D^2-a^2\right)\left[\left(D^2-a^2\right)^2-QD^2\right]W=-a^2R^2W. \tag{3.1}$$

The absence of terms in C from this expression make it apparent that hyperbolic heat-flow effects do not impact on the solutions for stationary convection, and results in this instance are equivalent to those found using the Fourier heat-flow law. Indeed, equation (3.1) is identical to the classical eigenfunction problem for magnetised conducting fluids described by Chandrasekhar [3–5], whose results we now summarise briefly for later comparison with the oscillatory modes.

For fixed boundary conditions (2.13b) equation (3.1) must be solved numerically (see §5), and in this section we consider free boundaries (2.13a) only, in which case all even derivatives of W vanish at $z=1,0$, and $W(z)$ can be written in terms of a Fourier series comprising odd terms

$$W(z)=\sum_{n=1}^{\infty}W_n,\quad \text{where}\quad W_n=A_n\sin(n\pi z) \tag{3.2}$$

is the nth mode weighted by the constant coefficient A_n. Thence, equation (3.1) yields a Rayleigh number corresponding to the nth mode [3]

$$R_a(n)=R^2(n)=\frac{\Lambda_n(\Lambda_n^2+n^2\pi^2Q)}{a^2},\quad \text{where}\quad \Lambda_n\equiv n^2\pi^2+a^2,\quad \text{or equivalently} \tag{3.3}$$

$$\hat{R}=\frac{(1+x_n)}{x_n}\left[(1+x_n)^2+\hat{Q}\right],\quad \text{with}\quad \hat{R}\equiv\frac{R_a}{n^4\pi^4},\quad x_n\equiv\frac{a^2}{n^2\pi^2},\quad \hat{Q}\equiv\frac{Q}{n^2\pi^2}, \tag{3.4}$$

as a set of further '$n^2\pi^2$' normalisations. Further, solving $\left.\frac{\partial\hat{R}}{\partial x_n}\right|_{x_c}=0$, where $x_c(n)=a_c^2/n^2\pi^2$ is the critical wavenumber at which $\hat{R}$ is minimised (both a and x_n shall be referred to as the wavenumber), one obtains the critical Rayleigh number R_c for the onset of stationary convection

$$\hat{R}_c\equiv\hat{R}(x_c;n),\quad \text{or alternatively}\quad R_c\equiv R_a(a_c;n),\quad \text{where}\quad 2x_c^3+3x_c^2-1=\hat{Q}. \tag{3.5}$$

Observe from equation (3.3) that, for some given wavenumber a, the sequence of Rayleigh numbers $\{R_a(a;n)\}$ increases monotonically with n. Thus, the absolute critical Rayleigh number and wavenumber are those associated with the lowest mode $n=1$, in which case we have [3]

$$R_c(Q)\to\begin{cases}\frac{27\pi^4}{4}, & \text{as}\quad Q\to 0,\\ \pi^2Q, & \text{as}\quad Q\to\infty,\end{cases}\quad \text{with}\quad a_c(Q)\to\begin{cases}\frac{\pi}{\sqrt{2}}, & \text{as}\quad Q\to 0,\\ \left(\frac{1}{2}\pi^4Q\right)^{1/6}, & \text{as}\quad Q\to\infty.\end{cases} \tag{3.6}$$

We shall adopt the convention that *unless the mode number n is stated in a parameter's argument explicitly, or made clear by the context, expressions shall be quoted assuming $n=1$ throughout.*

4. Oscillatory Convection

We now consider the onset of instability in system (2.11) *via* oscillatory modes of convection, i.e., overstability. The existence of such modes in magnetised fluids obeying a Fourier type heat-flow law have long been known. Indeed, Chandrasekhar discusses the topic at length in *Hydrodynamic and Hydromagnetic Stability* [3], and—for the geometry considered here—it has been shown that overstability is the preferred form of convection provided Q exceeds some threshold value $Q_T(\mathcal{P}_1, \mathcal{P}_2)$ [20]. Similarly, we have demonstrated recently that overstability can also occur in unmagnetised fluids subject to a Cattaneo-Christov heat-flow model [1], and represents the preferred form of instability whenever the Cattaneo number C exceeds a threshold value $C_T(\mathcal{P}_1)$. When both magnetic fields and Cattaneo effects are simultaneously present, therefore, we expect the manner of onset of instability to depend in some way on each of the parameters C, Q, $\mathcal{P}_1$ and $\mathcal{P}_2$. In general, the nature of this dependence will be rather complicated, but as we now discuss, some of its features may be uncovered by appropriate simplifications. Note that our attention shall be restricted initially to free boundary conditions (2.13*a*), since fixed boundary conditions (2.13*b*) will require a numerical treatment (see §5).

For the problem of oscillatory convection, we assume that σ may in general be complex, and define γ such that [3]

$$\sigma \equiv i\gamma. \tag{4.1}$$

Since we deal in this section with free boundary conditions (2.13*a*), we find as for stationary convection that all even derivatives of $W(z)$ vanish at the boundaries, so that substitution of the Fourier expansion (3.2) into the general eigenfunction function problem (2.12) yields a relationship for the nth mode

$$\begin{aligned}
&\left\{\Lambda_n\left(\Lambda_n - 2C\mathcal{P}_1^2\gamma^2\right)\left(\Lambda_n^2 - \gamma^2\mathcal{P}_2 + n^2\pi^2 Q\right) - \gamma^2\Lambda_n^2\mathcal{P}_1\left(1+\mathcal{P}_2\right)\right\} \\
&+\left\{\mathcal{P}_1\Lambda_n\left(\Lambda_n^2 - \gamma^2\mathcal{P}_2 + n^2\pi^2 Q\right) + \Lambda_n^2\left(\Lambda_n - 2C\mathcal{P}_1^2\gamma^2\right)\left(1+\mathcal{P}_2\right)\right\}\sigma \\
&\quad = \left\{\Lambda_n - 2C\mathcal{P}_1\mathcal{P}_2\gamma^2\right\}a^2 R_a + \left\{2C\mathcal{P}_1\Lambda_n + \mathcal{P}_2\right\}a^2 R_a\sigma,
\end{aligned} \tag{4.2}$$

where we have written terms in σ^2 as $-\gamma^2$ according to the definition (4.1). For given frequency, and wavenumber a, this expression may be used to calculate the corresponding Rayleigh number R_a required for the nth mode to satisfy system (2.11). Our task here is to find the minimum Rayleigh number for which such solutions are physical, and, since R_a must be real, this places constraints on the relationship between the complex quantities σ and γ. Proceeding in the tradition of Chandrasekhar [3], we shall study the onset of convection *via* a purely oscillatory mode, i.e., $\Re\{\sigma\} = 0$, and assert γ to be real. Hence, by comparing real and imaginary parts in equation (4.2) we may eliminate terms in σ to obtain the following quadratic in γ^2

$$\begin{aligned}
&\gamma^4\left\{4C^2\mathcal{P}_1^3\mathcal{P}_2^2\right\} + \gamma^2\left\{\mathcal{P}_2^2(1+\mathcal{P}_1) - 2C\mathcal{P}_1\left(\mathcal{P}_2^2\Lambda_n - 2C\mathcal{P}_1^2\left[\Lambda_n^2 + n^2\pi^2 Q\right]\right)\right\} \\
&\quad - \left\{2C\mathcal{P}_1\Lambda_n\left[\Lambda_n^2 + n^2\pi^2 Q\right] - \Lambda_n^2(\mathcal{P}_1+1) + (\mathcal{P}_2 - \mathcal{P}_1)n^2\pi^2 Q\right\} = 0.
\end{aligned} \tag{4.3}$$

The standard solution to the quadratic implies two solutions for γ^2; however, it may be shown that our requirement $\gamma^2 > 0$ means one of these solutions can be discarded, and we have

$$\begin{aligned}
\gamma^2 = &\frac{1}{8C^2\mathcal{P}_1^3\mathcal{P}_2^2}\Bigg\{ -\left\{\mathcal{P}_2^2(1+\mathcal{P}_1[1-2C\Lambda_n]) + 4C^2\mathcal{P}_1^3\left[\Lambda_n^2 + n^2\pi^2 Q\right]\right\} \\
&+ \sqrt{\left\{\mathcal{P}_2^2(1+\mathcal{P}_1[1-2C\Lambda_n]) - 4C^2\mathcal{P}_1^3\left[\Lambda_n^2 + n^2\pi^2 Q\right]\right\}^2 + 16C^2\mathcal{P}_1^3\mathcal{P}_2^2(1+\mathcal{P}_2)}\Bigg\}.
\end{aligned} \tag{4.4}$$

Thus, by the imaginary part of equation (4.2), we arrive at a general expression for the Rayleigh number

$$R_a = \left(\frac{\Lambda_n^3[1+\mathcal{P}_1+\mathcal{P}_2] + \mathcal{P}_1\Lambda_n n^2\pi Q - \mathcal{P}_1\Lambda_n\left[(2C\mathcal{P}_1\Lambda_n+\mathcal{P}_2)+2C\mathcal{P}_1\mathcal{P}_2\Lambda_n\right]\gamma^2}{a^2(2C\mathcal{P}_1\Lambda_n+\mathcal{P}_2)} \right). \tag{4.5}$$

For given values of $\mathcal{P}_1$, $\mathcal{P}_2$, Q, and C, and for given mode number n, this equation defines the Rayleigh number R_a as a function of the wavenumber a. By calculating the minimum value of R_a, one may then determine the critical wavenumber a_c, and thence critical Rayleigh numbers R_c for the onset of instability by oscillatory convection.

In the case that $\{\mathcal{P}_2^2(1+\mathcal{P}_1[1-2C\Lambda_n]) + 4C^2\mathcal{P}_1^3[\Lambda_n^2+n^2\pi^2 M]\} \geq 0$,[2] careful examination of equation (4.4) reveals that a necessary condition for our requirement $\gamma^2 > 0$ to hold is

$$\mathcal{P}_2 - \mathcal{P}_1[1-2C\Lambda_n] > 0. \tag{4.6}$$

Comparing this expression with Chandrasekhar's results for magnetised convection [3], we see that one effect of the Cattaneo-Christov heat-flux is to relax the classical necessary condition for oscillatory convection, i.e., that $\mathcal{P}_2 > \mathcal{P}_1$, according to the transformation $\mathcal{P}_1 \to \mathcal{P}_1[1-2C\Lambda_n]$. Furthermore, when $(1+\mathcal{P}_1[1-2C\Lambda_n]) > 0$ it may also be shown that $\gamma^2 > 0$ is possible only if

$$\hat{Q} > \hat{Q}^*, \quad \text{where} \quad \hat{Q}^* = \left(\frac{1+\mathcal{P}_1[1-2C\Lambda_n]}{\mathcal{P}_2 - \mathcal{P}_1[1-2C\Lambda_n]} \right)(1+x_n)^2; \tag{4.7}$$

this too can be compared with Chandrasekhar's classical results [3], namely the lower bound $\hat{Q} > \hat{Q}^* = \frac{(1+\mathcal{P}_1)}{(\mathcal{P}_2-\mathcal{P}_1)}(1+x_n)^2$, where again we observe the transformation $\mathcal{P}_1 \to \mathcal{P}_1[1-2C\Lambda_n]$. Here the balance between C and $\mathcal{P}_1$ is perhaps to be expected given that increases in $\mathcal{P}_1(\nu)$ correspond to 'stiffening' of the fluid by virtue of the viscosity dependence ν, whereas hyperbolic heat-flow effects in C impart a kind of elasticity [1].

Equation (4.5) represents the most general solution to the problem of overstability in a magnetised fluid with Cattaneo heat-flow effects as described by system (2.11). The complexity of the general form does, however, somewhat obscures the nature of the $\mathcal{P}_1$, $\mathcal{P}_2$, Q, and C dependences, and it is therefore expedient to consider whether there is simpler means of exploring the combined effects of Cattaneo heat-flow and magnetic field without recourse to exact solution. To this end we now follow in the tradition of Chandrasekhar's treatment of combined magnetic and rotational effects, and explore conditions for which the magnetic Prandtl number $\mathcal{P}_2$ can be neglected [3]. Such a move has two advantages: first, the value of $\mathcal{P}_2$ will typically be small in any case (at least when compared to $\mathcal{P}_1$); and second, in the classical problem of magnetised thermal convection, overstability is only possible when $\mathcal{P}_2 \neq 0$ [3]. Thus, in what follows we can be certain that oscillatory convection arises solely as a result of hyperbolic heat-flow effects, that is, we will essentially be studying the impact of magnetic fields on the overstable solutions determined in our original context [1].

In principle one may remove terms in $\mathcal{P}_2$ by binomial expansion of our previous results in the limit $\mathcal{P}_2 \to 0$; however, in practice it is more straightforward to return to equations (4.2) after setting $\mathcal{P}_2 = 0$. Thence, solving for the frequency of oscillation, we obtain

$$\gamma^2 = \frac{1}{4C^2\mathcal{P}_1^3}\Big[\mathcal{P}_1\left(2C\Lambda_n - 1\right) - r_n\Big], \quad \text{where} \quad r_n(\Lambda_n, Q) \equiv \frac{\Lambda_n^2}{\Lambda_n^2 + n^2\pi^2 Q} \leq 1, \tag{4.8}$$

with equality holding in the definition of r_n in the limit $Q \to 0$, in which case we recover the frequency of oscillation for unmagnetised Cattaneo convection [1]. Thus, since we require $\gamma^2 > 0$, oscillatory convection is only possible when $2C\Lambda_n > 1$. Similarly, we find that the Rayleigh

[2] The situation is more involved when $\{\mathcal{P}_2^2(1+\mathcal{P}_1[1-2C\Lambda_n]) + 4C^2\mathcal{P}_1^3[\Lambda_n^2+n^2\pi^2 M]\} < 0$, and requires analysis in greater detail than would be appropriate here.

number for the nth mode is given by

$$R_a = \frac{\Lambda_n(\mathcal{P}_1 + r_n) + 2C\mathcal{P}_1^2(\Lambda_n^2 + n^2\pi^2 Q)}{4C^2\mathcal{P}_1^2 a^2}, \tag{4.9}$$

from which one recovers the unmagnetised Cattaneo convection result as $Q \to 0$ and $r_n \to 1$ [1]. Indeed, observe that one my write this equation as

$$R_a = R_S + R_Q, \quad \text{where} \tag{4.10}$$

$$R_S = \frac{1}{4C^2\mathcal{P}_1^2 a^2}\left[(\mathcal{P}_1 + 1)\Lambda_n + 2C\mathcal{P}_1^2\Lambda_n^2\right], \quad \text{and} \quad R_Q = \frac{n^2\pi^2 Q}{4C^2\mathcal{P}_1^2 a^2}\left[2C\mathcal{P}_1^2 - \frac{r_n}{\Lambda_n}\right], \tag{4.11}$$

are the Rayleigh numbers for oscillatory convection with the Cattaneo-Christov model of our original study [1], and a magnetic correction term respectively. Writing the Rayleigh number in this way helps to separate the unmagnetised hyperbolic heat-flow effects from new aspects introduced by the Chandrasekhar number Q; however, it is also useful for considering the mode number for which the Rayleigh numbers are smallest. In particular, in our earlier context we showed that for given a, C, and $\mathcal{P}_1$ that the sequence of Rayleigh numbers $\{R_S(n)\}$ is strictly increasing with n [1]. Similarly, it may be shown that the sequence $\{s_n\} = \{r_n/\Lambda_n\}$ is strictly decreasing, and thus that $\{R_Q(n)\}$, is also strictly increasing with n. In this way, we have that the minimum Rayleigh number for oscillatory convection may be found by selecting $n = 1$. However, in what follows it shall often be convenient to work in the '$n^2\pi^2$' normalisations, and for this reason we introduce

$$\hat{\gamma} \equiv \frac{\gamma}{n^2\pi^2}, \quad \text{and} \quad \hat{C} \equiv n^2\pi^2 C, \tag{4.12}$$

such that the oscillation frequency and Rayleigh numbers may be expressed as

$$\hat{\gamma}^2 = \frac{1}{4\hat{C}^2\mathcal{P}_1^3}\left[\mathcal{P}_1\left[2\hat{C}(1 + x_n) - 1\right] - r_n\right], \quad \text{where} \quad r_n(x_n, \hat{Q}) = \frac{(1 + x_n)^2}{(1 + x_n)^2 + \hat{Q}}, \tag{4.13}$$

$$\text{and} \quad \hat{R} = \frac{(1 + x_n)}{4\hat{C}^2\mathcal{P}_1^2 x_n}\left[\left(\mathcal{P}_1 + \frac{(1 + x_n)^2}{(1 + x_n)^2 + \hat{Q}}\right) + 2\hat{C}\mathcal{P}_1^2\left((1 + x_n) + \frac{\hat{Q}}{(1 + x_n)}\right)\right]. \tag{4.14}$$

It will be seen from these expressions that while $\mathcal{P}_2 \to 0$ represents a useful simplifying assumption, the nature of our various dependences mean that the solution is non-trivial. Several features remain to be shown: first, the conditions whereby oscillatory convection is a permitted solution, i.e., $\gamma^2 > 0$; second, the critical values at which the Rayleigh number $\hat{R}$ is minimised; and third, the range of parameters for which oscillatory convection is the preferred manner of onset of instability. We consider these problems in the following subsections.

(a) Cut-off Wavenumber for Oscillatory Solutions

To determine the preferred manner of onset of instability, we shall need to explore those regions of parameter space where the critical Rayleigh number for oscillatory convection $R_c^{(o)}$ is less than that for stationary convection $R_c^{(c)}$. However, our requirement that the oscillation frequency be real means that we must also confirm $\gamma^2 > 0$ whenever $R_c^{(o)} < R_c^{(c)}$, and this places restrictions on the wavenumber for oscillatory modes. Indeed, by equation (4.13) we have both the necessary, and the necessary and sufficient conditions

$$(1 + x_n) > \frac{1}{2\hat{C}}, \quad \text{and} \quad \mathcal{P}_1\left[2\hat{C}(1 + x_n) - 1\right] > r_n = \frac{(1 + x_n)^2}{(1 + x_n)^2 + \hat{Q}} < 1 \tag{4.15}$$

respectively. Such inequalities prompt us to define a cut-off wavenumber $x_n = x_n^*$ for which $\hat{\gamma}^2(x_n^*) = 0$, and by equation (4.13) we see that x_n^* will be given by solutions to the cubic

$$P_*(y_n) = 2\mathcal{P}_1\hat{C}y_n^3 - (\mathcal{P}_1 + 1)y_n^2 + 2\mathcal{P}_1\hat{C}\hat{Q}y_n - \mathcal{P}_1\hat{Q} = 0. \quad \text{where} \quad y_n \equiv (1 + x_n), \tag{4.16}$$

Hence, permitted solutions obtain whenever $P_*(y_n) > 0$. Since $P_*(y_n)$ has either one or three real positive roots y_n^*, which we can label $y_n^* = y_w$ (one root) and $y_n^* \in \{y_u, y_v, y_w\}$ (three roots, $y_u < y_v < y_w$), for permitted solutions we require either $y_w < y_n$ (if one root), or (if three roots) either $y_u < y_n < y_v$ or $y_w < y_n$. While it is not possible to solve equation (4.16) meaningfully, it may be shown that $\left(\mathcal{P}_1, \hat{C}\sqrt{\hat{Q}}\right)$ parameter space is divided into regions for which either one or three roots obtain by the curve

$$C_{Q\pm} = \frac{\sqrt{2}(\mathcal{P}_1+1)^{3/2}}{\sqrt{\mathcal{P}_1(1+20\mathcal{P}_1-6\mathcal{P}_1^2) \mp \mathcal{P}_1(1-8\mathcal{P}_1)^{3/2}}}, \quad \text{where} \quad C_Q \equiv \hat{C}\sqrt{\hat{Q}}, \tag{4.17}$$

with the triple root solution represented by the cusp co-ordinate $(\mathcal{P}_1, C_Q) = (\frac{1}{8}, \frac{3\sqrt{2}}{2})$. Thus, provided we restrict ourselves to $\mathcal{P}_1 \geq \frac{1}{8}$, in which case only one root (or a triple root) is possible, only one cut-off wavenumber obtains, and we can make general statements regarding the possibility of solutions (see section (c)). Indeed, notice that if the cut-off x_n^* is less than or equal to zero, i.e., $y_c \leq 1$, then all wavenumbers will yield permitted oscillatory solutions.

(b) Critical Rayleigh Number for Oscillatory Modes

By equation (4.14) we have that for given $\mathcal{P}_1$, Q, and C, the Rayleigh number is a function of the wavenumber x_n. Thus, proceeding as we did in section 3, differentiating $\hat{R}(x_n)$ with respect to x_n and solving $\partial R_n/\partial x_n|_{x_c} = 0$, we may obtain an expression for the critical wavenumber x_c for the nth mode, and thence critical the Rayleigh number $\hat{R}_c \equiv \hat{R}(x_c)$. Indeed, defining

$$x_S(n) \equiv \left[1 + \frac{(\mathcal{P}_1+1)}{2\hat{C}\mathcal{P}_1^2}\right]^{1/2}, \tag{4.18}$$

we find that the critical x_c will be given by solutions to

$$(x_S^2 - x_c^2 + \hat{Q}) = \frac{\hat{Q}}{2\hat{C}\mathcal{P}_1^2}\left(\frac{2x_c(1+x_c)^2 + [(1+x_c)^2 + \hat{Q}]}{[(1+x_c)^2+\hat{Q}]^2}\right). \tag{4.19}$$

This equation represents a sextic polynomial in x_c and is not amenable to further solution. In addition, numerical investigation reveals that under some conditions, particularly very small values of the Prandtl number, more than one root with $x_c > 0$ is permitted. Consequently, care must be taken when computing the critical Rayleigh number to ensure that the correct value of x_c is used such that $\hat{R}(x_c)$ represents an absolute (global) minimum, and, indeed, that such a solution is permitted with $x_n^* < x_c$.

Though it is not possible to solve for x_c exactly, we can calculate the asymptotic limits, and in this way reveal at least some of the system qualities analytically. As expected, in the limit of low Chandresekhar number we recover the unmagnetised results from our original context [1], whereas at high Q we find

$$\lim_{Q\to\infty} x_c = \sqrt{\hat{Q}}, \quad \lim_{Q\to\infty} \hat{R}_c = \frac{\sqrt{\hat{Q}}}{\hat{C}}, \quad \text{and} \quad \lim_{Q\to\infty} \hat{\gamma}_c^2 = \frac{\sqrt{\hat{Q}}}{2\hat{C}\mathcal{P}_1^2}, \tag{4.20}$$

where the absolute critical values correspond to the case where $n = 1$ (provided that these solutions are permitted, see §(a)). In the absence of closed form expressions, these limits allow us to compare the Rayleigh numbers for stationary and oscillatory convection, which we may denote $R_c^{(c)}$ and $R_c^{(o)}$ respectively, when Q is large. Notice in particular that with $n = 1$ we have $R_c^{(o)} \to \pi^2(\sqrt{Q}/C)$ as $Q \to \infty$, whereas for stationary convection we found $R_c^{(c)} \to \pi^2 Q$ as $Q \to \infty$ (see equation (3.6)), i.e., that hyperbolic heat-flow effects lead to much weaker inhibition of onset of instability when the Chandrasekhar number is large. As we shall discuss in the following section, one consequence of this is that we expect oscillatory convection to represent the preferred form of instability at large Q even if the Cattaneo number C is itself small. Indeed, by solving

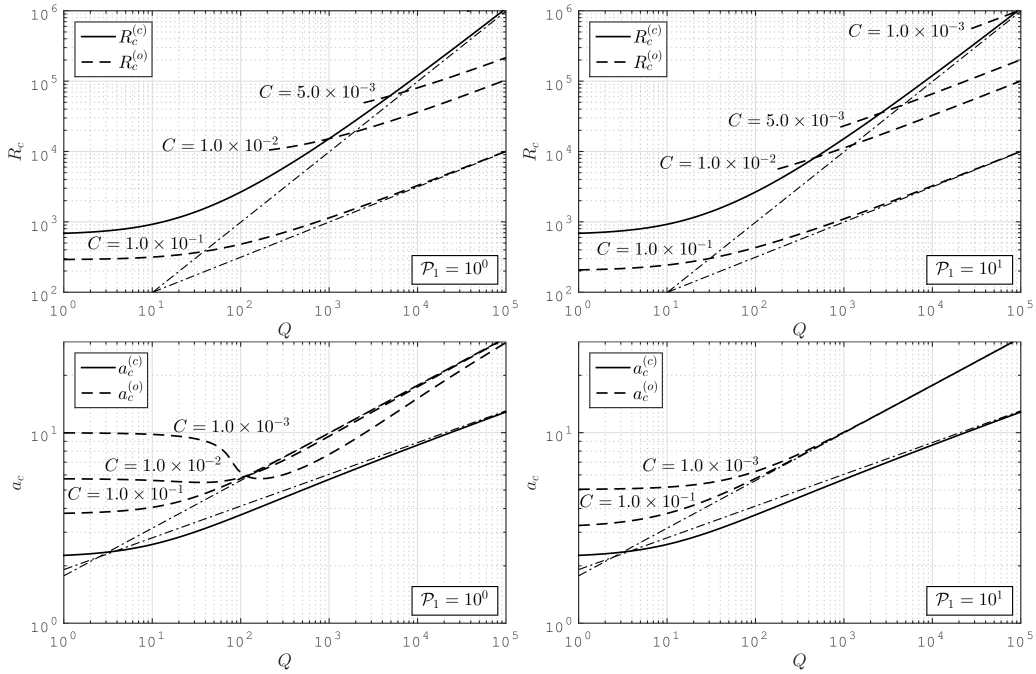

Figure 1. Critical Rayleigh numbers R_c (top), wavenumbers a_c (bottom), and illustrative asymptotic limits (dash-dotted curves) plotted as a function of Q for free boundary conditions with $\mathcal{P}_1 = 1$ (left) and $\mathcal{P}_1 = 10$ (right), and several values of Cattaneo number (as labelled). Critical Rayleigh number curves for each form of convention intersect at the points $(Q, C_T(Q))$, where C_T is the *threshold Cattaneo number* corresponding to Q (see §(c)). Here we have selected the critical mode $n = 1$, while only those portions of the R_c curves corresponding to permitted solutions ($\gamma_c^2 > 0$) are shown; full curves are shown for a_c indicating their non-trivial Q dependence. In these calculations the sextic (4.19) yields only one physical root for the critical wavenumber.

equation (4.19) numerically for given $\mathcal{P}_1$, C and Q, in figure 1 we compare values for the Rayleigh numbers $R_c^{(c)}$ and $R_c^{(o)}$, and their corresponding critical wave-numbers $a_c^{(c)}$ and $a_c^{(o)}$ respectively, for a range of Q. These plots suggest that for each value of the Cattaneo number C, there is some threshold Chandrasekhar number $Q_T(C, \mathcal{P}_1)$ beyond which $R_c^{(c)} > R_c^{(o)}$ and the preferred manner of onset of instability changes from stationary to oscillatory convection (see §4(c) below). [This threshold may be conceived of equivalently as a threshold Cattaneo number $C_T(Q, \mathcal{P}_1)$.] Note that at fixed Q, the behaviour at large Cattaneo number C is

$$\lim_{C\to\infty} x_c = \sqrt{1+\hat{Q}}, \quad \lim_{C\to\infty} \hat{R}_c = \frac{1}{\hat{C}}\left[1+\sqrt{1+\hat{Q}}\right], \quad \text{and} \quad \lim_{C\to\infty} \gamma_c^2 = \frac{1+\sqrt{1+\hat{Q}}}{2\hat{C}\mathcal{P}_1^2}; \tag{4.21}$$

while at low C we have that solutions are non-physical, with $x_c \to x_S \to \infty$, and $\hat{R}_c \to \infty$.

(c) Preferred Manner of Onset of Instability

We now consider the problem of determining the preferred manner by which instability occurs, using the superscripts '(c)', and '(o)' for quantities corresponding to stationary and oscillatory convection respectively. Without closed form expressions for either of the critical Rayleigh numbers, such an investigation is non-trivial; however, some analytical progress can be made by considering the more general formulae for R_a as a function of wave-number. First, observe by equations (3.4) and (4.14) that the Rayleigh numbers for stationary and oscillatory convection,

$\hat{R}^{(c)}$ and $\hat{R}^{(o)}$, are related by the expression

$$\hat{R}^{(o)} = \hat{R}^{(c)} - \frac{\hat{\gamma}^2\mathcal{P}_1}{x_n}\left[2\hat{C}\mathcal{P}_1(1+x_n)^2 + (1+x_n) + 2\hat{C}\mathcal{P}_1\hat{Q}\right]. \tag{4.22}$$

Hence, in the $(x_n, \hat{R})$ plane we have that the solutions for oscillatory convection branch off from those for stationary convection when $\gamma^2 = 0$, i.e., at the point $x_n = x_n^*$ (recall that x_n^* is unique for $\mathcal{P}_1 > \frac{1}{8}$, see §(a)). It then follows that for a given wavenumber x_n, oscillatory convection is the preferred manner for onset of instability whenever solutions for $\hat{R}^{(o)}(x_n)$ are permitted ($\gamma^2 > 0$). If we now consider the critical wave-number for stationary convection $x_c^{(c)}$, then we find by equations (3.5) and (4.13) that for given $\mathcal{P}_1$ and Q

$$\hat{\gamma}^2(x_c^{(c)}) > 0 \quad \text{and thus} \quad \hat{R}^{(o)}(x_c^{(c)}) < \hat{R}_c^{(c)} \quad \text{whenever} \quad \hat{C} > \frac{\left(1+2\mathcal{P}_1 x_c^{(c)}\right)}{4\mathcal{P}_1 x_c^{(c)}\left(1+x_c^{(c)}\right)}; \tag{4.23}$$

that is, we have the sufficient condition on C for oscillatory convection to represent the preferred manner of instability (for which only the critical mode $n=1$ is relevant)

$$C > C_S(\mathcal{P}_1, Q) \equiv \frac{\left(\pi^2 + 2\mathcal{P}_1(a_c^{(c)})^2\right)}{4\mathcal{P}_1(a_c^{(c)})^2\left(\pi^2 + (a_c^{(c)})^2\right)}, \quad \text{with} \quad \lim_{Q\to\infty} C_S = \frac{1}{(4\pi^4 Q)^{1/3}}, \tag{4.24}$$

where C_S is defined here as the *sufficient Cattaneo number*, and $a_c^{(c)}(Q)$ is given by the solution to the cubic (3.5) (see figure 2). Note that for unmagnetised conditions $C_S(\mathcal{P}_1) = (1+\mathcal{P}_1)/3\pi^2\mathcal{P}_1$.

Inequality (4.24) is a sufficient and necessary condition for $R^{(o)}(x_c^{(c)}) < R_c^{(c)} \equiv R^{(c)}(x_c^{(c)})$, but it is not a necessary condition for $R_c^{(o)} \equiv R^{(o)}(x_c^{(o)}) < R_c^{(c)}$. The non-trivial dependence of $x_c^{(o)}$ (the critical wavenumber for oscillatory convection) on the Cattaneo number means that it is difficult to make general statements concerning the C dependence of $R_c^{(o)}$; however, results from our earlier context [1], in addition to our asymptotic limits (4.21), and substantial numerical investigation, suggest that $R_c^{(o)}$ is a strictly decreasing function of C for given $\mathcal{P}_1$ and Q. Indeed, our asymptotic limits indicate that for given $\mathcal{P}_1$ and Q there is some *threshold Cattaneo number* C_T beyond which ($C > C_T$) onset of instability will be as oscillatory convection, where C_T is defined such that (cf. figure 1)

$$R_c^{(o)}(\mathcal{P}_1, Q, C_T) = R_c^{(c)}(\mathcal{P}_1, Q), \tag{4.25}$$

In this way we may define both threshold wave-numbers and gyration frequencies corresponding to $R_c^{(o)}(C_T)$, which we denote $a_T = a_c^{(o)}(C_T)$ and $\gamma_T = \gamma_c^{(o)}(C_T)$ respectively (see §6, figure 4).

Whilst it does not look easy to solve for C_T in general, as before we can gain some insight into the values of the threshold parameters by considering the behaviour at limiting values of Q. When Q is small, these values are simply those found in our earlier context [1], with

$$C_T^{3/2}\left[C_T^{1/2} - \sqrt{\frac{2\pi^2}{R_c^{(c)}}}\right] = \frac{(\mathcal{P}_1+1)}{4R_c^{(c)}\mathcal{P}_1^2}, \quad \text{where} \quad R_c^{(c)} = \frac{27\pi^4}{4}. \tag{4.26}$$

Conversely, the task of determining asymptotic limits for $Q\to\infty$ is not straightforward; however, in essence one finds that

$$\lim_{Q\to\infty} C_T = \left(\frac{\phi_T(1+b_T)}{4\sqrt{b_T}}\right)\frac{1}{\pi Q^{1/2}} = 0, \quad \text{with} \quad \phi_T(\mathcal{P}_1) \equiv \frac{4b_T^2}{\mathcal{P}_1^2(1+b_T)^3(1-b_T)}, \tag{4.27}$$

where $b_T \equiv b_T(\mathcal{P}_1)$ is a strictly increasing function of $\mathcal{P}_1$ given by the solutions to a sextic

$$\mathcal{P}_1^3 + \mathcal{P}_1^2\frac{b_T(3-b_T)}{(1+b_T)(1-b_T)} - \left(\frac{2b_T}{(1+b_T)^2}\right)^2\frac{b_T}{(1-b_T)^2} = 0, \tag{4.28}$$

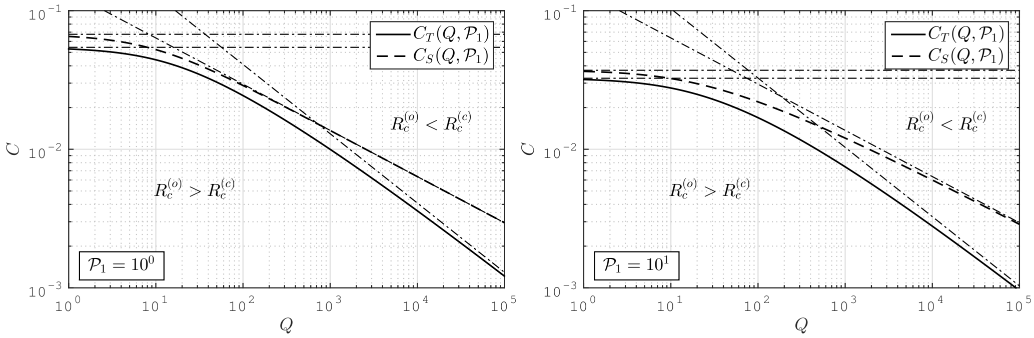

Figure 2. The sufficient Cattaneo number C_S (dashed curves), threshold Cattaneo number C_T (solid curves), and asymptotes (dash-dotted curves) plotted as a function of Q, with $\mathcal{P}_1 = 1$ (left) and $\mathcal{P}_1 = 10$ (right). For $C > C_S$ we have by equation (4.22) and inequality (4.23) that oscillatory convection will be the preferred manner of onset of instability, with $R_a^{(o)}(a_c^c) < R_c^{(c)} \equiv R_a^{(c)}(a_c^c)$; conversely, when $C < C_T$, stationary convection will be the preferred manner of onset of instability, with $R_c^{(o)} > R_c^{(c)}$. Between the two curves C_T and C_S we expect instability to occur as overstability ($R_c^{(o)} < R_c^{(c)}$); however, whilst numerical investigation is supportive of this claim, since we have not demonstrated that *all* oscillatory solutions are permitted in this region, such an expectation is unproven. Data in this figure corresponds to free boundary conditions, with values for C_T computed by solving the intersection problem of equation (4.25) numerically (cf. §5 and figure 3). A subset of the data used to produce these plots is given in table 1.

and it may be shown that $\phi(\mathcal{P}_1) \in (2,4)$ and $b_T \in (0,1)$, with

$$\phi_T(\mathcal{P}_1) \to \begin{cases} 2, & \text{as} \quad \mathcal{P}_1 \to 0, \\ 4, & \text{as} \quad \mathcal{P}_1 \to \infty, \end{cases} \quad \text{and} \quad b_T(\mathcal{P}_1) \to \begin{cases} \mathcal{P}_1 \to 0, & \text{as} \quad \mathcal{P}_1 \to 0, \\ \left(1 - \frac{1}{4\mathcal{P}_1^2}\right) \to 1, & \text{as} \quad \mathcal{P}_1 \to \infty. \end{cases} \tag{4.29}$$

Similarly, the asymptotes for threshold wave-numbers and gyration frequencies are

$$\lim_{Q\to\infty} a_T = \left(b_T \pi^2 Q\right)^{1/4} = \infty, \quad \text{and} \tag{4.30}$$

$$\lim_{Q\to\infty} \gamma_T^2 = \frac{4\pi^2 b_T}{\mathcal{P}_1^3 (1+b_T)^2 \phi_T^2} \left\{ \mathcal{P}_1 \left[\frac{1}{2}(1+b_T)\phi_T - 1 \right] - \frac{b_T}{b_T+1} \right\} Q = \infty. \tag{4.31}$$

With recourse to both the sufficient C_S and threshold C_T Cattaneo numbers, (Q, C) parameter space may be divided into regions where each form of onset of instability is preferred, as we have done in figure 2. For $C > C_S$ onset of instability is as oscillatory convection, whereas for $C < C_T$, onset of instability is as stationary convection. There thus remains a region $C_T < C < C_S$ where the exact manner of onset of instability is in some sense uncertain, because while substantial numerical investigation suggests that all oscillatory solutions with $C > C_T$ are permitted, this has not been proven in general. Notice from these plots that C_T appears to decrease with $\mathcal{P}_1$, in agreement with the results for unmagnetised Catteneo convection in our earlier context [1].

5. Numerical Solution

For fixed boundary conditions, such as those given by equation (2.13*b*), system (2.11) must be solved numerically, and to this end we employ the Chebyshev-τ method described by Dongerra *et al.* [10] as adopted in our earlier context [1], and for which the z-coordinate is transformed such that our problem is defined on $z \in (-1, 1)$. The basic principle of this method is to eliminate terms with derivatives higher than D^2, so that here it is necessary for us to propose a novel numerical scheme by introducing the auxiliary variable $\chi(z)$ defined such that

$$\chi(z) f(x,y) e^{\sigma t} = \nabla^2 w. \tag{5.1}$$

Thence, operating on equation (2.11*d*) with D, and substituting the result into equation (2.11*a*), system (2.11) may be written in the augmented form

$$(D^2-a^2)W-\chi=0, \tag{5.2a}$$

$$-Qa^2W+[(D^2-a^2)-Q]\chi-a^2R\Theta=\sigma(\chi-QDK), \tag{5.2b}$$

$$\mathcal{P}_2DW+(D^2-a^2)K=\sigma\mathcal{P}_2K, \tag{5.2c}$$

$$(D^2-a^2)\Theta+S=-2\sigma C\mathcal{P}_1S, \tag{5.2d}$$

$$-RW+S=-\sigma\mathcal{P}_1\Theta. \tag{5.2e}$$

Further, by expanding our quantities $\Phi\in\{W,\chi,K,\Theta,S\}$ as Chebyshev polynomials $T_n(z)$ weighted by constant coefficients ϕ_n, *viz*

$$\Phi(z)=\sum_{n=0}^{N-1}\phi_nT_n(z), \quad \text{with } N \text{ even}, \tag{5.3}$$

system (5.2) may be approximated to an arbitrary level of precision by the eigenvalue problem

$$A_{ij}\mathrm{x}_j=\sigma B_{ij}\mathrm{x}_j, \quad i,j=0,\ldots,(5N-1) \tag{5.4}$$

where x_j is a vector of length $5N$ comprising the ϕ_n, i.e.,

$$\mathrm{x}_j=(w_0,\ldots,w_{N-1},\chi_0,\ldots,\chi_{N-1},k_0,\ldots,k_{N-1},\theta_0,\ldots,\theta_{N-1},q_0,\ldots,q_{N-1})^{\mathrm{T}}, \tag{5.5}$$

and $A_{ij}\equiv A_{ij}(a,R,\mathcal{P}_2,Q)$ and $B_{ij}\equiv B_{ij}(C,\mathcal{P}_1,\mathcal{P}_2,Q)$ are $5N\times 5N$ matrices with constant coefficients defined in equation (5.9) below. Notice that on the boundaries of the domain the Chebyshev polynomials satisfy

$$T_n(\pm1)=(\pm1)^n \quad \text{and} \quad T_n'(\pm1)=(\pm1)^{n-1}n^2, \tag{5.6}$$

so that by equations (2.13*b*), for rigid surfaces we can form the set of expanded fixed boundary conditions (B.C.s)

$$W(\pm1)=0\begin{cases}\text{B.C. 1:} & w_0+w_2+w_4+\cdots+w_{N-2}=0,\\ \text{B.C. 2:} & w_1+w_3+w_5+\cdots+w_{N-1}=0,\end{cases} \tag{5.7a}$$

$$DW(\pm1)=0\begin{cases}\text{B.C. 3:} & 2^2w_2+4^2w_4+6^2w_6+\cdots+(N-2)^2w_{N-2}=0,\\ \text{B.C. 4:} & w_1+3^2w_3+5^2w_5+\cdots+(N-1)^2w_{N-1}=0,\end{cases} \tag{5.7b}$$

$$K(\pm1)=0\begin{cases}\text{B.C. 5:} & k_0+k_2+k_4+\cdots+k_{N-2}=0,\\ \text{B.C. 6:} & k_1+k_3+k_5+\cdots+k_{N-1}=0,\end{cases} \tag{5.7c}$$

$$\Theta(\pm1)=0\begin{cases}\text{B.C. 7:} & \theta_0+\theta_2+\theta_4+\cdots+\theta_{N-2}=0,\\ \text{B.C. 8:} & \theta_1+\theta_3+\theta_5+\cdots+\theta_{N-1}=0,\end{cases} \tag{5.7d}$$

which may also be used in the case of free surfaces provided one replaces B.C.s 3 & 4 with

$$D^2W(\pm1)=\chi(\pm1)=0\begin{cases}\text{B.C. 9:} & \chi_0+\chi_2+\chi_4+\cdots+\chi_{N-2}=0,\\ \text{B.C. 10:} & \chi_1+\chi_3+\chi_5+\cdots+\chi_{N-1}=0,\end{cases} \tag{5.8}$$

respectively (see equation (2.13*a*)). These boundary conditions are incorporated into the numerical solver at the stage of constructing the matrices A_{ij} and B_{ij}. Indeed, if we use the superscript notation $M^{\mathrm{X,Y}}$ to indicate a matrix M modified such that its penultimate row is over-written with an equation for boundary condition B.C. X, and its final row over-written with an equation for boundary condition B.C. Y, and the notation $M^{0,0}$ to indicate a matrix M modified such that its final two rows have been over-written with zeros, then A_{ij} and B_{ij} are given by

$$A_{ij}=\begin{pmatrix}(4D^2-a^2I)^{1,2} & (-I)^{0,0} & 0 & 0 & 0\\ (-Qa^2I)^{3,4/0,0} & (4D^2-a^2I-QI)^{0,0/9,10} & 0 & (-a^2RI)^{0,0} & 0\\ (2\mathcal{P}_2D)^{0,0} & 0 & (4D^2-a^2I)^{5,6} & 0 & 0\\ 0 & 0 & 0 & (4D^2-a^2I)^{7,8} & (I)^{0,0}\\ -RI & 0 & 0 & 0 & I\end{pmatrix}$$

$$\text{and}\quad B_{ij}=\begin{pmatrix}0 & 0 & 0 & 0 & 0\\ 0 & (I)^{0,0} & (-2QD)^{0,0} & 0 & 0\\ 0 & 0 & (\mathcal{P}_2I)^{0,0} & 0 & 0\\ 0 & 0 & 0 & 0 & (-2C\mathcal{P}_1I)^{0,0}\\ 0 & 0 & 0 & -\mathcal{P}_1I & 0\end{pmatrix} \tag{5.9}$$

respectively. Here each of the coefficients represents a $N\times N$ sub-matrix block, with $I_{ns}=\delta_{ns}$, $n,s=0,\ldots,(N-1)$ as the identity tensor, and D_{ns} as the *differentiation matrix* defined by Dongerra *et al.* [10]. The choice of boundary conditions in the second row block (rows $N+1$ to $2N$) corresponds to fixed/free surfaces.

For given parameters a, C, $\mathcal{P}_1$, $\mathcal{P}_2$, and Q (though here we set $\mathcal{P}_2=0$) , the matrix system (5.4) may be solved iteratively to determine a value of $R=\sqrt{R_a}$ such that $\Re\{\sigma\}=0$. Critical values are then obtained by minimising $R_a(a)=R^2$ with respect to wavenumber a. In this way both stationary ($\gamma=0$) and oscillatory ($\gamma\neq 0$) solutions may be found for a range of $\mathcal{P}_1$, Q and C, and one may produce critical curves of the kind shown in figure 1. [For free boundary conditions, this procedure provides a convenient means of testing the accuracy of the numerical solver.]

When Q and $\mathcal{P}_1$ are fixed, the Rayleigh number for stationary convection $R_c^{(c)}$ is constant, whereas the Rayleigh number for oscillatory convection $R_c^{(o)}$ varies with Cattaneo number. Thus, by computing $R_c^{(o)}(C)$ for a range of C, one obtains an intersection problem for the threshold Cattaneo number $C_T(Q)$; in particular, we may bound this threshold to the interval $C_T\in(C_l,C_u)$, where $R_c^{(o)}(C_l)>R_c^{(c)}>R_c^{(o)}(C_u)$, and thus determine C_T numerically to within some chosen tolerance (this process is described in greater detail in our earlier context [1]). Here we do so for Prandtl numbers $\mathcal{P}_1=1$ and $\mathcal{P}_1=10$, and 10001 values of Chandrasekhar number in the interval $Q\in[10^{-1},10^{+4}]$ (see table 1, & figure 3). Note that numerical accuracy is most strongly influenced by the resolution for scans over wavenumber a when calculating a_c, though by employing a suitably fine mesh, we are able to determine the threshold parameters a_T, γ_T, and C_T to within $\pm 0.1\%$. Other considerations on numerical accuracy include truncation of the Chebychev polynomial expansion; however, we find (up to at least six significant figures) that changing the number of Chebyshev polynomials from $N=40$ to $N=50$ does not affect computed values of R_c, and all numerical data is thus quoted assuming $N=40$.

6. Comments on Transitions Between Forms of Convection

Observe from table 1 that the threshold Cattaneo number $C_T(Q)$ is of comparable magnitude given either free or fixed boundary conditions, while in both instances C_T decreases with increasing Prandtl number $\mathcal{P}_1$. As expected, this behaviour is consistent with our previous results for oscillatory convection with the Cattaneo-Christov heat-flow law in unmagnetised conditions [1]. Furthermore, a comparison between figures 2 and 3 suggests that for fixed boundary conditions we can expect similar asymptotic dependence to that found for free boundaries, i.e., $C_T\propto 1/\sqrt{Q}$ as $Q\to\infty$. Crucially, therefore, the asymptotic behaviour of C_T means that for sufficiently large Chandrasekhar numbers Q, oscillatory convection can represent the preferred form of instability even when the Cattaneo number C itself is small.

The $C_T(Q)$ curves displayed in figures 2 and 3 divide (Q,C) parameter space into regions where either stationary or oscillatory convection represents the preferred manner of onset of instability, and can be equivalently interpreted as a threshold Chandrasekhar number $Q_T(C)$

for given C. Thus, for fixed $\mathcal{P}_1$ and C, transitions between the two forms of convection can be triggered by increasing Q beyond a threshold $Q_T(C)$. As shown in figure 4, transitions of this kind are marked by a discontinuous shift in the critical wavenumber $a_c^{(c)}$ to larger critical wavenumber a_T, and thus smaller convection cells (as we found for unmagnetised convection with hyperbolic heat-flow effects in our earlier context [1]). Such a result has important consequences for investigating the verisimilitude for the Cattaneo-Christov heat-flow law, as marked transitions between forms of overall system behaviour can in principle be controlled by appropriate 'tuning' of the Chandrasekhar number Q.

In essence, the $C_T \propto 1/\sqrt{Q}$ scaling of the threshold Cattaneo number as $Q \to \infty$ may be expected from the corresponding fixed C scalings of the critical Rayleigh numbers $R_c^{(c)} \to \pi^2 Q$ and $R_c^{(o)} \to \pi\sqrt{Q}/C$ given by limits (3.6) and (4.20) respectively. As noted by Chandrasekhar, the absence of $\mathcal{P}_1$ from these expressions is an indication that dissipation of energy at high Q is dominated by Joule heating (as controlled by the resistivity η) rather than viscosity ν [3]. In the case of stationary convection, this interpretation of quasi-inviscid convection at large Q supports

Q	Free Boundary Conditions				Fixed Boundary Conditions			
	$\mathcal{P}_1=1$		$\mathcal{P}_1=10$		$\mathcal{P}_1=1$		$\mathcal{P}_1=10$	
	C_T	a_T	C_T	a_T	C_T	a_T	C_T	a_T
0	0.05439	4.086	0.03254	3.268	0.03214	5.175	0.02149	4.834
$10^{-1.0}$	0.05424	4.090	0.03247	3.275	0.03213	5.176	0.02148	4.837
$10^{-0.8}$	0.05415	4.092	0.03243	3.280	0.03209	5.179	0.02146	4.839
$10^{-0.6}$	0.05401	4.095	0.03236	3.286	0.03206	5.181	0.02146	4.841
$10^{-0.4}$	0.05380	4.100	0.03226	3.297	0.03203	5.183	0.02144	4.845
$10^{-0.2}$	0.05346	4.108	0.03211	3.313	0.03197	5.187	0.02140	4.852
$10^{+0.0}$	0.05294	4.120	0.03186	3.338	0.03187	5.195	0.02133	4.861
$10^{+0.2}$	0.05215	4.139	0.03149	3.377	0.03171	5.204	0.02125	4.878
$10^{+0.4}$	0.05100	4.168	0.03095	3.437	0.03149	5.222	0.02112	4.902
$10^{+0.6}$	0.04935	4.213	0.03016	3.524	0.03111	5.248	0.02091	4.941
$10^{+0.8}$	0.04711	4.279	0.02908	3.651	0.03059	5.287	0.02060	5.000
$10^{+1.0}$	0.04421	4.377	0.02766	3.828	0.02977	5.350	0.02015	5.088
$10^{+1.2}$	0.04070	4.517	0.02590	4.067	0.02866	5.444	0.01949	5.221
$10^{+1.4}$	0.03675	4.713	0.02385	4.375	0.02718	5.581	0.01862	5.409
$10^{+1.6}$	0.03257	4.979	4.75861	1.333	0.02529	5.776	0.01748	5.669
$10^{+1.8}$	0.02839	5.329	0.01925	5.224	0.02309	6.048	0.01614	6.013
$10^{+2.0}$	0.02440	5.772	0.01690	5.773	0.02064	6.411	0.01461	6.455
$10^{+2.2}$	0.02074	6.314	0.01465	6.412	0.01814	6.875	0.01300	6.998
$10^{+2.4}$	0.01746	6.960	0.01255	7.147	0.01568	7.447	0.01138	7.653
$10^{+2.6}$	0.01459	7.714	0.01064	7.983	0.01339	8.139	0.009833	8.423
$10^{+2.8}$	0.01211	8.585	0.008949	8.931	0.01130	8.958	0.008387	9.312
$10^{+3.0}$	0.009992	9.579	0.007469	10.00	0.009450	9.904	0.007090	10.34
$10^{+3.2}$	0.008206	10.71	0.006193	11.21	0.007854	10.99	0.005939	11.50
$10^{+3.4}$	0.006712	11.99	0.005107	12.56	0.006475	12.23	0.004935	12.83
$10^{+3.6}$	0.005470	13.43	0.004190	14.09	0.005316	13.65	0.004077	14.32
$10^{+3.8}$	0.004443	15.05	0.003423	15.80	0.004343	15.25	0.003348	16.00
$10^{+4.0}$	0.003599	16.88	0.002787	17.72	0.003534	17.05	0.002738	17.91

Table 1. Numerically computed values for the threshold Cattaneo numbers C_T, and wave-numbers a_T, for both free and fixed boundary conditions. Threshold values are determined by solving the intersection problem of equation (4.25); for free boundaries this is done using the analytical expressions of Section 4, whereas for fixed boundaries the Chebychev-τ scheme described in Section 5 is employed. This table comprises a subset of the values used to produce figures 2, 3 and 4 for Chandrasekhar numbers in the range $Q \in [10^{-1}, 10^{+4}]$, with data quoted to within $\pm 0.1\%$ uncertainty.

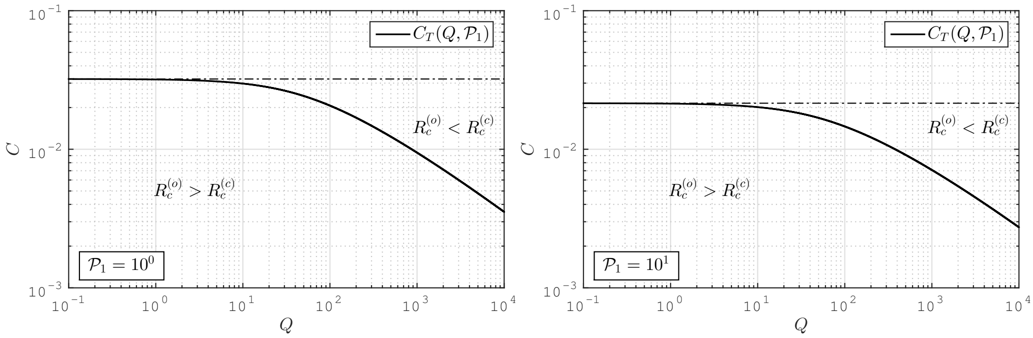

Figure 3. Threshold Cattaneo numbers $C_T(Q)$ (solid curve) as a function of Q for $\mathcal{P}_1 = 1$ (left) and $\mathcal{P}_1 = 10$ (right) in the case of fixed boundary conditions (dash-dotted curves represent $C_T(0)$). Above the curve we expect the preferred manner of instability to be oscillatory convection according to $R_c^{(o)} < R_c^{(c)}$, whereas below the curve stationary convection prevails with $R_c^{(o)} > R_c^{(c)}$ (cf. figure 2). A subset of the data used to produce these plots is given in table 1.

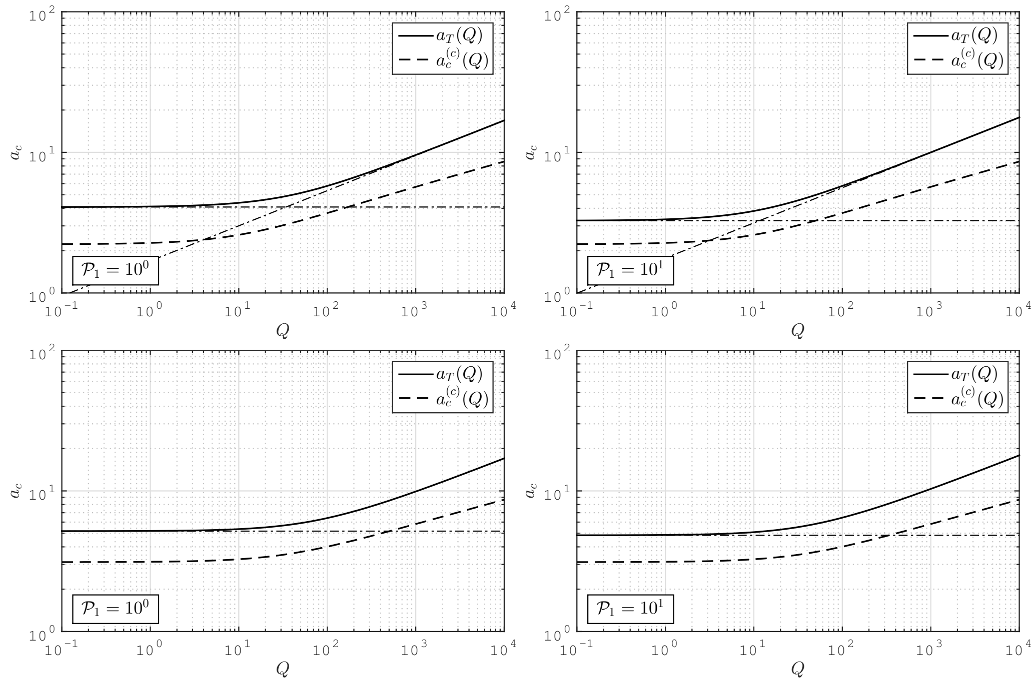

Figure 4. Threshold wavenumbers $a_T \equiv a_c^{(o)}(\mathcal{P}_1, C_T)$ for oscillatory convection ($C_T \equiv C_T(Q)$), and critical wavenumbers for stationary convection $a_c^{(c)}(Q)$ as a function of Chandrasekhar number Q for both free (top) and fixed (bottom) boundary conditions, and with Prandtl numbers $\mathcal{P}_1 = 1$ (left) and $\mathcal{P}_1 = 10$ (right). The dash-dotted curves are the asymptotic limits given by equation (4.30), and the values for $Q = 0$ as computed in our earlier context [1].

the narrowing of convection cells as shown in figure 4 [3]. For oscillatory convection the physical basis of such asymptotic limits is less clear, though it may be possible to obtain further insight by considering the geometry of the convection cells, and their impact on wave-motion, as part of a more advanced non-linear analysis. Indeed, the presence of $\mathcal{P}_1$ dependence in the limit for the threshold Cattaneo number is at first glance rather puzzling (see equation (4.27)). However, this feature is understood by observing that the high Q limits on $R_c^{(o)}$ assume fixed C, whereas (when solving for the threshold Cattaneo number) C_T vanishes like $C_T \propto 1/\sqrt{Q}$, modifying the leading order terms in $R_c^{(o)}(Q, C_T)$, and thence solutions to $R_c^{(o)}(Q, C_T) = R_c^{(c)}(Q)$ as $Q \to \infty$.

7. Conclusions

We have studied the impact of hyperbolic heat-flow effects on the problem of thermal convection in a magnetised conducting fluid layer heated from below, by replacing the classical Fourier law with the Catteneo-Christov heat-flow formulation [1,3,7,25], and in this way developed a linear theory for Catteo-driven oscillatory convection in the presence of an impressed magnetic field. In the case of free boundary conditions, analytical approaches yield critical Rayleigh numbers for the onset of instability by either stationary or oscillatory convection, which we denote $R_c^{(c)} \equiv R_a^{(c)}(a_c^{(c)})$ and $R_c^{(o)} \equiv R_a^{(o)}(a_c^{(o)})$ respectively, corresponding to critical wavenumbers $a_c^{(c)}$ and $a_c^{(o)}$, and the lowest mode number $n = 1$ (see §2, 3, & 4). For stationary convection, absence of a gyration component to the mode frequency ($\sigma \equiv i\gamma = 0$) means that effects arising from the Cattaneo number C vanish, and we recover the classical result (i.e., that obtained using the more usual Fourier law) of suppressed onset of instability by the impressed field, with $R_c^{(c)}$ dependent on the Chandrasekhar number Q, and exhibiting asymptotic behaviour $R_c^{(c)} \to \pi^2 Q$ as $Q \to \infty$ [3]. However, for oscillatory convection ($\gamma \neq 0$) we find that the Rayleigh number $R_a^{(o)}(\mathcal{P}_1, \mathcal{P}_2, Q, C)$ becomes a more complicated function of Q, C, and both the Prandtl and magnetic Prandtl numbers $\mathcal{P}_1$ and $\mathcal{P}_2$ respectively; this result is expected given that in the classical, Fourier heat-flow formulation of the problem [3], oscillatory convection can obtain provided $\mathcal{P}_2 > \mathcal{P}_1$ [20].

The complexity of the exact expression for $R_a^{(o)}$ precludes direct solution; however, following in the tradition of Chandrasekhar's treatment of combined rotational and magnetic effects [3], the impact of Cattaneo terms on magnetised convection has been furthered by considering conditions under which $\mathcal{P}_2$ can be neglected, in which case overstability would be classically forbidden. Thus, by setting $\mathcal{P}_2 \to 0$ we have derived expressions for the critical Rayleigh number $R_c^{(o)}$ which indicate weaker inhibition of instability by the impressed field when compared to stationary convection (§4(b)), as exemplified in the high Q limit for which $R_c^{(o)} \to \pi\sqrt{Q}/C$. Indeed, investigation of (a, R_a) parameter space guarantees that Rayleigh numbers $R_a^{(o)}(a_c^{(c)})$ for oscillatory convection are lower than the critical Rayleigh numbers for stationary convection $R_c^{(c)} = R_a^{(c)}(a_c^{(c)})$ whenever C exceeds a value C_S, where $C_S(\mathcal{P}_1, Q)$ can be computed exactly, and has limiting behaviour $C_S \propto 1/Q^{1/3}$ (§4(a) & 4(b)). More strongly, we have argued that there exists a threshold Cattaneo number $C_T(Q)$ beyond which ($C > C_T$) the preferred manner onset of instability switches from stationary to oscillatory convection, and in this way divided (Q, C) parameter space into regions were each form of convection prevails (§4(c)). For large Chandrasekhar numbers Q, this threshold scales as $C_T \propto 1/\sqrt{Q}$, meaning that oscillatory modes are preferred even when C itself is small.

For fixed boundary conditions the impact of Cattaneo terms on magnetised convection must be studied using computational approaches, and here we have done so by developing a novel numerical scheme following expansion of the problem into Chebychev polynomials (§5). In so doing, we have investigated the transition from stationary to oscillatory convection for Chandrasekhar numbers in the range $Q \in [10^{-1}, 10^{+4}]$, recovering similar behaviour to that found for free boundaries, including discontinuous shifts in the mode wavenumber, and apparent asymptotic dependence for the threshold Cattaneo number $C_T \propto 1/\sqrt{Q}$ as $Q \to \infty$ (see §6). One possible area of future work, therefore, would be to confirm this dependence at high Q; indeed, the fixed boundary problem should be amenable to further analytical treatment when $Q \to \infty$ by means of boundary later solutions. By using $N = 40$ polynomials, threshold values for both the Cattaneo number C_T, and oscillatory wavenumber $a_T \equiv a_c^{(o)}(C_T)$ have been tabulated here for the more general cases ($Q \in [10^{-1}, 10^{+4}]$) to within $\pm 0.1\%$ (see table 1).

Both the theoretical and numerical analyses presented here represent a substantial development of our earlier work on Cattaneo-Christov heat-flow effects on thermal convection

[1]; nevertheless, while we have succeeded in deriving a number of new analytical results and asymptotic limits, several theoretical questions remain. For example, the linear approach adopted here precludes an investigation of convection cell geometry, making it difficult to establish reasons for the discontinuous shift in wavenumber between the stationary and oscillatory regimes. Of particular importance, however, is establishing a physical basis for the $C_T \propto 1/\sqrt{Q}$ dependence of the threshold Cattaneo number at large Q. In our earlier context we described how hyperbolic heat-flow effects impart a kind of elasticity to the fluid, allowing thermal disturbances to propagate as waves, and it is thus intuitive that Cattaneo terms should promote the onset of oscillatory modes [1]. Perhaps one reason for the $C_T \propto 1/\sqrt{Q}$ dependence, therefore, is coupling between thermal and Alfvén waves; indeed, for classical oscillatory convection (forbidden here due to vanishing $\mathcal{P}_2$), Chandrasekhar shows that the Alfvén wave-speed determines the effective gyration frequency γ [3], and it seems plausible that an analogous effect may be acting here. Given the importance of the threshold Cattaneo number C_T for determining the preferred manner of onset of instability, these questions warrant further investigation in future studies.

Taken together, our results for both free and fixed boundary conditions imply that hyperbolic heat-flow effects can have a profound impact on magnetised thermal convection, with significant lowering of thresholds for onset of instability as exemplified by our solutions for oscillatory modes. Indeed, our analysis emphasises the significance of Cattaneo terms in determining overall system behaviour, beyond simply the speed at which thermal signals propagate [2,14,15,19]. In particular, modifications to the Chandrasekhar number Q at fixed C and $\mathcal{P}_1$ can trigger bifurcations in the preferred manner of onset of instability by pushing the system beyond the $C_T(Q)$ threshold. That an impressed magnetic field inhibits Cattaneo driven overstability [1] is expected given that magnetisation is known to suppress classical stationary convection [3]; however, such inhibition is far less dramatic than that of the stationary modes, meaning that Cattaneo terms lead to a comparably enhanced effect on the onset of convection overall. Crucially, therefore, our study implies that even relatively weak hyperbolic heat-flow effects have the potential to become pronounced in the presence of large magnetic fields.

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